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Characters of Bounded $\widehat{\mathfrak{sl}(2)}$ Modules

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1. INTRODUCTION

Throughout this paper we assume that the base field is $\mathbb{C}$.

1.1

Let $\mathfrak{g}$ be an affine Lie algebra and let $\mathfrak{h}$ be its Cartan subalgebra. By [5], $\mathfrak{h}$ admits a non-degenerate bilinear form $(\cdot, \cdot)$, which is positive semi-definite on $\mathbb{Q}\pi$, where π stands for the set of simple roots of $\mathfrak{g}$. That form defines an isomorphism ψ: $\mathfrak{h}^* \to \mathfrak{h}$. For each $\alpha \in \pi$, $\dim \mathfrak{g}_\alpha = 1$ and $\mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha}$ generates a Lie subalgebra $\mathfrak{s}_\alpha$ of $\mathfrak{g}$ which is isomorphic to $\mathfrak{sl}(2)$. There is a unique positive imaginary root $\delta \in \mathfrak{h}^*$ such that $(\delta, \alpha) = 0$ for all $\alpha \in \pi$ and all the imaginary roots of $\mathfrak{g}$ are integer multiples of δ. We call $K = \psi(\delta)$ the canonical central element.

A $\mathfrak{g}$-module M is a *weight module* if it decomposes into a direct sum of its $\mathfrak{h}$ weight subspaces $M_\nu : \nu \in \mathfrak{h}^*$. Set $\Omega(M) = \{\nu \in \mathfrak{h}^* : M_\nu \neq 0\}$. We say that a weight module M is *admissible* if $\dim M_\nu < \infty$ for all $\nu \in \mathfrak{h}^*$ and define its formal character by

$$\operatorname{ch} M = \sum_{\nu \in \Omega(M)} (\dim M_\nu) e^\nu.$$

A $\mathfrak{g}$-module M is said to be *integrable* [5, Sect. 3.6] if it is a direct sum (possibly infinite) of finite dimensional $\mathfrak{s}_\alpha$-submodules.

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DEFINITION [3]. A weight module M is said to be bounded if there exists $n \in \mathbb{N}$ such that $\operatorname{Re}(\nu, \nu) \leq n$ for all $\nu \in \Omega(M)$.

Notice that a bounded module is necessarily integrable, as for any $\nu : M_\nu \neq 0$ and $\alpha \in \pi$, an upper bound on $\operatorname{Re}(\nu + k\alpha, \nu + k\alpha)$ imposes some bound on $|k|$, whence $x_{\pm\alpha} \in \mathfrak{g}_{\pm\alpha}$ act locally nilpotently on M. By [3, 4], any simple bounded module is admissible. This result can also be obtained from [1], but we use the presentation of [4], which we find more convenient. Suppose that M is simple and bounded. Then $\Omega(M) \subset \nu_0 + \mathbb{Z}\pi$; hence $(\nu, \delta) = (\nu_0, \delta)$ for all $\nu \in \Omega(M)$ and K acts on M by a scalar. If that scalar is non-zero the situation resembles the highest weight theory. If K acts trivially then M can be shown [1, 3, 4] to be a simple subquotient of a tensor product of loop modules. In this work we shall only consider the latter case.

1.2

If V is a vector space, set $\mathscr{L}(V) := V \otimes_{\mathbb{C}} \mathbb{C}[t, t^{-1}]$. Let $\mathfrak{g}_0$ be a finite dimensional simple Lie algebra. By [5, Theorem 7.4], an untwisted affine algebra can be realized as $\hat{\mathscr{L}}(\mathfrak{g}_0) := \mathscr{L}(\mathfrak{g}_0) \oplus \mathbb{C}K \oplus \mathbb{C}D$ with the Lie structure defined by

$$[x \otimes t^n, y \otimes t^m] = [x, y] \otimes t^{n+m} + n\delta_{n+m,0}(x, y)_0 K,$$
$$[D, x \otimes t^n] = nx \otimes t^n,$$

for all $x, y \in \mathfrak{g}_0$, $m, n \in \mathbb{Z}$, where $(\cdot\,,\cdot)_0$ is a multiple of the Killnig form of $\mathfrak{g}_0$. If $\mathfrak{g}_0$ is of type X_N, then $\mathfrak{g} = \hat{\mathscr{L}}(\mathfrak{g}_0)$ is said to be of type $X_N^{(1)}$. Let $\mathfrak{g}'$ denote its derived Lie algebra $[\mathfrak{g}, \mathfrak{g}]$. Then $\mathfrak{g}' = \mathscr{L}(\mathfrak{g}_0) \oplus \mathbb{C}K$. Finally, a subalgebra $\mathfrak{h} = \mathfrak{h}_0 \oplus \mathbb{C}K \oplus \mathbb{C}D$, where $\mathfrak{h}_0$ stands for a Cartan subalgebra of $\mathfrak{g}_0$, is a Cartan subalgebra of $\mathfrak{g}$.

Let V be a finite dimensional $\mathfrak{g}_0$-module. For each $a \in \mathbb{C}^\times$, V can be made into a $\mathfrak{g}'$-module called an *evaluation module* by setting $(x \otimes P).v = P(a)xv$, for all $x \in \mathfrak{g}_0$, $P \in \mathbb{C}[t, t^{-1}]$, $v \in V$, and $KV = 0$. It can be shown that every finite dimensional simple $\mathfrak{g}'$-module is isomorphic to a quotient of a tensor product of such evaluation modules. The latter is simple if the evaluation points are distinct. In particular, its character is just the product of characters of the corresponding finite dimensional $\mathfrak{g}_0$-modules.

Similarly the action of $\mathfrak{g}$ on a tensor product of loop modules $\mathscr{L}(V_0) \otimes \cdots \otimes \mathscr{L}(V_{k-1}) \cong V_0 \otimes \cdots \otimes V_{k-1} \otimes \mathbb{C}[t_0^{\pm 1}, \ldots, t_{k-1}^{\pm 1}]$ where $V_0, \ldots, V_{k-1}$ are $\mathfrak{g}_0$-modules is given by

$$\begin{aligned} (x \otimes P)(v \otimes Q) &= \sum_{r=0}^{k-1} x^{(r)}(v) \otimes P(t_r)Q, \\ D(v \otimes Q) &= v \otimes \sum_{r=0}^{k-1} t_r \frac{\partial Q}{\partial t_r} \end{aligned} \tag{1.1}$$

for all $x \in \mathfrak{g}_0$, $v \in V_0 \otimes \cdots \otimes V_{k-1}$, $P \in \mathbb{C}[t, t^{-1}]$, and $Q \in \mathbb{C}[t_0^{\pm 1}, \ldots, t_{k-1}^{\pm 1}]$, and for all $y \in \mathfrak{g}_0$,

$$y^{(r)} = \mathrm{id}^{\otimes r} \otimes y \otimes \mathrm{id}^{\otimes k-r-1} \in \mathrm{End}(V_0 \otimes \cdots \otimes V_{k-1}). \tag{1.2}$$

If $k > 1$ our modules have infinite dimensional weight spaces, so the simple quotients are always non-trivial and it is not at all obvious what their characters should be.

1.3

Here we compute these characters for $\mathfrak{g}$ of type $A_1^{(1)}$ in the case corresponding to the tensor product of evaluation modules at distinct points. This is quite non-trivial and involves some interesting data on modified Euler functions $\phi_n(d)$ (cf. Corollary 4.2) introduced according to [2] in 1902 by von Sterneck. Indeed our main result is

PROPOSITION. *Assume that $\mathfrak{g}$ is of type $A_1^{(1)}$ and let M be a generic simple bounded module (cf. 3.1) in which the canonical central element K acts trivially. Then there exist $m, \ell \in \mathbb{N}^+$ such that up to translation of a multiple of δ*

$$\mathrm{ch}\, M = \frac{1}{m} e^{m\ell\omega} \sum_{k=0}^{m\ell} \left(\sum_{n \in \mathbb{Z}} \sum_{d \mid m, k} \phi_n(d) \binom{m\ell/d}{k/d} e^{n\delta} \right) e^{-k\alpha}, \tag{1.3}$$

where α is a simple root of $\mathfrak{g}$ and ω is the corresponding fundamental weight.

The strategy of our proof is as follows. We first eliminate all t variables except one (Lemma 2.4). Then we construct a matrix by applying lowering operators defined in 3.2 on the highest weight spaces. We find a basis in which each column of our matrix has at least one non-zero entry (3.3, Lemma 4.1). This gives an upper bound on its rank (Corollary 4.2). Since the evaluation points are distinct, it follows from Lemma 2.5 that our module is a direct sum of simples. Through a summation formula (Lemma 4.3) for the modified Euler functions we deduce in 4.4 that this is also a lower bound.

2. PRELIMINARIES

2.1

Retain the notation of 1.2 and let $\mathfrak{g}$ be an affine algebra of type $A_1^{(1)}$ which is also denoted as $\widehat{\mathfrak{sl}(2)}$. In that case $\mathfrak{g}_0 \cong \mathfrak{sl}(2)$. Let $\{x_\alpha, h_\alpha, x_{-\alpha}\}$ be the canonical generators of $\mathfrak{g}_0$. Then $\mathfrak{g}$ is a linear span of $x_{\pm\alpha}(n) := x_{\pm\alpha} \otimes t^n$, $h(n) := h_\alpha \otimes t^n$, for all $n \in \mathbb{Z}$. Furthermore, $\mathfrak{h} := \mathbb{C}h_\alpha \oplus \mathbb{C}K \oplus$

$\mathbb{C}D$ is a Cartan subalgebra of $\mathfrak{g}$. Define $\alpha, \delta \in \mathfrak{h}^*$ by $\alpha(K) = \alpha(D) = 0$, $\delta(K) = \delta(h_\alpha) = 0$, $\alpha(h_\alpha) = 2$, $\delta(D) = 1$. Then $\alpha|_{\mathbb{C}h_\alpha}$ is the positive root of $\mathfrak{g}_0$, δ is the imaginary root of 1.1, and the $\mathfrak{h}$-roots of $\mathfrak{g}$ are $\{n\delta \pm \alpha, n\delta : n \in \mathbb{Z}\}$. Let $\mathfrak{k}$ denote a Heisenberg subalgebra of $\mathfrak{g}$, $\mathfrak{k} = \mathbb{C}K \oplus \mathfrak{k}_0$, where $\mathfrak{k}_0 := \bigoplus_{n \in \mathbb{Z}} \mathbb{C}h(n)$. Define $\mathfrak{g}^{\pm} = \bigoplus_{n \in \mathbb{Z}} \mathbb{C}x_{\pm\alpha}(n) = \mathbb{C}x_{\pm\alpha} \otimes \mathbb{C}[t, t^{-1}]$ and set $\mathfrak{p}^+ = \mathfrak{k}_0 \oplus \mathfrak{h} \oplus \mathfrak{g}^+$.

Set $Q^{(1)}(n) = h(n)$ and, following [4], define inductively

$$\begin{aligned} Q^{(m)}&(n_1, \dots, n_m) \\ &:= \sum_{i=1}^{m} h(n_i) Q^{(m-1)}(n_1, \dots, \hat{n}_i, \dots, n_m) \\ &\quad - 2 \sum_{1 \le i < j \le m} Q^{(m-1)}\big(n_i + n_j, n_1, \dots, \hat{n}_i, \hat{n}_j, \dots, n_m\big), \quad (2.1) \end{aligned}$$

where $\hat{n}$ means omitting that argument. The following proposition, due to [4], justifies the definition of $Q^{(m)}(n_1, \dots, n_m)$.

PROPOSITION. *Let W be a simple $U(\mathfrak{p}^+)$-module in which $\mathfrak{g}^+$ acts by zero and $h(0)$ acts by m, $m \ge 1$. Then the induced module $M = U(\mathfrak{g}) \otimes_{U(\mathfrak{p}^+)} W$ admits an integrable quotient if and only if*

$$Q^{(m+1)}(n_0, \dots, n_m) W = 0, \qquad \forall n_0, \dots, n_m \in \mathbb{Z}. \qquad (2.2)$$

Again, if V is a $U(\mathfrak{g})$-module admitting a $\mathfrak{p}^+$-submodule W that satisfies $\mathfrak{g}^+ W = 0$, $x_{-\alpha}^{m+1} W = 0$, then (2.2) *holds.*

2.2

Let ω be the fundamental weight of $\mathfrak{g}_0$ and let $V(\omega)$ be the two-dimensional simple $\mathfrak{g}_0$-module. Then $L(\omega) = \mathscr{L}(V(\omega))$ admits a natural structure of a $\mathfrak{g}$-module. Set $\mathfrak{T} := \mathbb{C}[t_0^{\pm 1}, \dots, t_{m-1}^{\pm 1}]$. For any $m \in \mathbb{N}^+$, $V = L(\omega)^{\otimes m}$ is a $\mathfrak{g}$-module with respect to the diagonal action. We identify V with $V(\omega)^{\otimes m} \otimes \mathfrak{T}$. Let v_ω be a highest weight vector of $V(\omega)$ and set $v_{-\omega} = x_{-\alpha} v_\omega$. Then W of Proposition 2.1 is just $H(V) := \mathbb{C}v_\omega^{\otimes m} \otimes \mathfrak{T}$, in which $h(n)$ acts by the symmetric function

$$p_n(\mathbf{t}) = p_n(t_0, \dots, t_{m-1}) = \sum_{i=0}^{m-1} t_i^n, \qquad n \in \mathbb{Z}. \qquad (2.3)$$

PROPOSITION [4]. *Let $m \ge 1$ be an integer.*

1. *The relations $Q^{(m+1)}(n_0, \dots, n_m) = 0 : n_i \in \mathbb{Z}$ are satisfied by taking $h(n) = p_n(\mathbf{t})$.*

2. *All the relations of the polynomial algebra $U(\mathfrak{k}_0)$ are obtained from item 1 by taking just $n_0 \in \mathbb{Z}$, $n_i = \pm 1$, for all $i > 0$.*

Let $\bar{V}$ be simple bounded module of type m; that is, m is minimal among the positive integers k such that $x_{-\alpha}^{k+1}\bar{V} = 0$. Then up to translation of a multiple of δ

3. *$\bar{V}$ is a quotient of $U(\mathfrak{g})H(V)$,*

4. *$\bar{V}$ is generated by $H(\bar{V})$, which is a quotient of $H(V)$ as a $U(\mathfrak{p}^+)$-module; $H(\bar{V})$ has one-dimensional weight spaces and is a simple $U(\mathfrak{p}^+)$-module.*

From now on if M is a simple bounded module we use $H(M)$ to denote the sum of its highest weight subspaces, which is itself a simple $U(\mathfrak{p}^+)$-module.

2.3

Extend ω to the dual of the Cartain subalgebra of $\mathfrak{g}$ by requiring $\omega(K) = \omega(D) = 0$ and observe that the $\mathfrak{h}$-weights of $H(V)$ are $m\omega + n\delta : n \in \mathbb{Z}$.

LEMMA [4]. *Let $\bar{V}$ be a simple bounded module of type m. Then $\Omega(H(\bar{V})) = \{m\omega + nr\delta : n \in \mathbb{Z}\}$ for some r dividing m.*

Proof. Since $\bar{V}$ is bounded, it is integrable. We may assume, without loss of generality, that $m\omega \in \Omega(\bar{V})$. Since $\bar{V}$ is of type m, $x_{-\alpha}^m \bar{V}_{m\omega} \neq 0$. Then $m\omega - m\alpha \in \Omega(\bar{V})$, whence by [5, Proposition 11.1], $m\omega - m\alpha + m(\delta + \alpha) = m\omega + m\delta \in \Omega(\bar{V})$.

Choose $s_1 > s_2$, $m\omega + s_i\delta \in \Omega(\bar{V})$ such that $r = s_1 - s_2$ is minimal. By the simplicity of $H(\bar{V})$, there exist $y_\pm \in U(\mathfrak{k})_{\pm r\delta}$ such that $y_+\bar{V}_{m\omega + s_2\delta} = \bar{V}_{m\omega + s_1\delta}$ and $y_-\bar{V}_{m\omega + s_1\delta} = \bar{V}_{m\omega + s_2\delta}$. Then $y_-y_+ \in U(\mathfrak{k}_0)$ and acts on $H(\bar{V})$ as a non-zero scalar. Since $U(\mathfrak{k}_0)$ is commutative, $y_+y_- = y_-y_+$ and we can choose it to be 1. Therefore, $H(\bar{V})$ is generated by $\mathbb{C}[y_-, y_+]$ over $\bar{V}_{m\omega}$ and $\Omega(H(\bar{V})) = m\omega + \mathbb{Z}r\delta$. In particular, $r \mid m$. ∎

DEFINITION. A simple bounded module $\bar{V}$ is said to be of type (m, r) if $\Omega(H(\bar{V})) = m\omega + \mathbb{Z}r\delta$.

2.4

Let $\mathfrak{S}$ denote the subalgebra of $\mathfrak{T}$ generated by $p_{\pm n}(\mathbf{t})$, $n = 1, \dots, m$. It identifies with the S_m-invariant subalgebra of $\mathfrak{T}$. By a theorem of Chevalley, $\mathfrak{T}$ is a free module over $\mathfrak{S}$ of rank $m!$ and we may write $\mathfrak{T} = \mathfrak{H} \otimes \mathfrak{S}$, where $\mathfrak{H}$ is any graded vector space complement of $\mathfrak{T}\mathfrak{S}_+$ in $\mathfrak{T}$. We can view $H(\bar{V})$ as an $\mathfrak{S}$-module. In particular, $H(\bar{V}) \cong \mathfrak{S}/\operatorname{Ann}_{\mathfrak{S}} H(\bar{V})$. Set $\mathfrak{A} = \operatorname{Ann}_{\mathfrak{S}} H(\bar{V})$ and $\mathfrak{a} = \mathfrak{T}\mathfrak{A}$. It follows that $\mathfrak{T}/\mathfrak{a}$ identifies with $m!$ copies of $H(\bar{V})$ up to a translation of weight spaces by integer multiples of δ.

Observe that the quotient $L(m\omega, \mathfrak{a}) := V(\omega)^{\otimes m} \otimes \mathfrak{T}/\mathfrak{a}$ of V inherits a $\mathfrak{g}$-module structure. We may view $H(\overline{V})$ as a $\mathfrak{p}^+$-submodule of $L(m\omega, \mathfrak{a})$ through the decomposition $\mathfrak{T}/\mathfrak{a} = \mathfrak{S}/\mathfrak{A} \oplus (\mathfrak{H}_+ \otimes \mathfrak{S}/\mathfrak{A})$. Therefore, $\overline{V}$ is isomorphic to a submodule of $L(m\omega, \mathfrak{a})$.

LEMMA. *Suppose that $\ell = m/d \in \mathbb{N}$ and $\overline{V}$ is a simple bounded module of type (m, d). Let $\mathfrak{b}$ be an ideal of $\mathfrak{T}$ generated by*

$$t_i - a_i t_0, \qquad a_i \in \mathbb{C}^\times, i = 1, \dots, m-1, \tag{2.4}$$

where

$$a_{rd+s} = a_{rd}\zeta^s, \qquad 0 \le r < \ell, 0 < s < d,$$

and ζ is a dth primitive complex root of unity. Then

$$\mathfrak{b} \cap \mathfrak{S} = \operatorname{Ann}_{\mathfrak{S}} H(\overline{V}). \tag{2.5}$$

Conversely, if an ideal $\mathfrak{b} \subset \mathfrak{T}$ satisfies (2.5) *then $\mathfrak{b}$ is of the form* (2.4) *up to a permutation of the indeterminates. In particular, there are precisely $m!$ ideals $\mathfrak{b} \subset \mathfrak{T}$ such that* (2.5) *holds.*

Proof. By Proposition 2.2, $H(\overline{V})$ is a simple $U(\mathfrak{k}_0)$-module with one-dimensional D-invariant subspaces. Therefore, the ideal $\mathfrak{A}$ is generated by

$$p_{-rd}(\mathbf{t})p_{rd}(\mathbf{t}) - c_r, \qquad c_r \in \mathbb{C}^\times, r = 1, \dots, \ell, \tag{2.6}$$

where the c_r are determined by the relations from Proposition 2.2 and

$$p_n(\mathbf{t}), \qquad n = \pm 1, \dots, \pm m, d \nmid n. \tag{2.7}$$

One has

$$p_n(\mathbf{t}) = \sum_{i=0}^{m-1} a_i^n t_0^n \pmod{\mathfrak{b}} = t_0^n \sum_{r=0}^{m/d-1} a_{rd}^n \sum_{s=0}^{d-1} \zeta^{sn}.$$

The inner sum equals d if $d \mid n$ and $(\zeta^{dn} - 1)/(\zeta^n - 1) = 0$ otherwise. It follows that $\mathfrak{A} \subset \mathfrak{b}$. Since $\mathfrak{A}$ is a maximal D-stable ideal of $\mathfrak{S}$, $\mathfrak{b} \cap \mathfrak{S} = \mathfrak{A}$.

For the converse, consider a polynomial $P(X) = (X - t_0) \cdots (X - t_{m-1}) \in \mathfrak{T}[X]$. Let $e_k(\mathbf{t})$, $0 \le k \le r$, denote the kth elementary symmetric polynomial in $t_0, \dots, t_{m-1}$ that is the coefficient of $(-1)^k X^k$ in $P(X)$ and set $e_k(\mathbf{t}) = 0$ if $k > m$. Recall the Newton identity

$$ke_k(\mathbf{t}) = \sum_{j=1}^{k} (-1)^{j-1} p_j(\mathbf{t}) e_{k-j}(\mathbf{t}), \qquad k \in \mathbb{N}^+. \tag{2.8}$$

One immediately concludes that $e_s(\mathbf{t}) = 0 \pmod{\mathfrak{b}}$, $1 \le s \le d-1$. For any k, $d \le k \le m-1$, write $k = rd + s$, $0 \le s \le d-1$. Again, by (2.8)

$$\begin{aligned}(rd+s)e_{rd+s}(\mathbf{t}) &= \sum_{j=1}^{rd+s} (-1)^{j-1} p_j(\mathbf{t}) e_{rd+s-j}(\mathbf{t}) \\ &= \sum_{j=1}^{r} (-1)^{jd-1} p_{jd}(\mathbf{t}) e_{(r-j)d+s}(\mathbf{t}) \qquad \pmod{\mathfrak{b}},\end{aligned}$$

whence, by induction on r, $e_k(\mathbf{t}) = 0 \pmod{\mathfrak{b}}$ if $d \nmid k$. Since $e_k(\mathbf{t})$ is a homogeneous polynomial of degree k, it follows that $e_k(\zeta^s\mathbf{t}) = e_k(\mathbf{t}) \pmod{\mathfrak{b}}$ for all $s \in \mathbb{Z}$. Therefore, the image $\overline{P}(X)$ of $P(X)$ in $(\mathfrak{T}/\mathfrak{a})[X]$ satisfies $\overline{P}(\zeta^s X) = \overline{P}(X)$. On the other hand, the roots of $\overline{P}(X)$ are images of $t_0, \dots, t_{m-1}$ in $\mathfrak{T}/\mathfrak{b}$. Therefore, $\zeta^s t_i = t_{\sigma(i)} \pmod{\mathfrak{b}}$ for some $\sigma \in S_m$. Thus, up to a permutation of indeterminates

$$t_{rd+s} = \zeta^s t_{rd} \pmod{\mathfrak{b}} \quad 0 < s < d, 0 \le r < \ell.$$

Now, for all $r = 1, \dots, \ell$,

$$p_{-rd}(\mathbf{t}) p_{rd}(\mathbf{t}) = md + d^2 \sum_{0 \le i < j < \ell} \left(\frac{t_{id}}{t_{jd}}\right)^{rd} + \left(\frac{t_{jd}}{t_{id}}\right)^{rd} = c_r \qquad \pmod{\mathfrak{b}}.$$

It follows that $t_{id}/t_{jd} \in \mathbb{C}^\times \pmod{\mathfrak{b}}$. ∎

The above lemma implies that $\mathfrak{a}$ is an intersection of all the ideals $\mathfrak{b}$ satisfying 2.5.

2.5

Due to Lemma 2.4, we can identify $V(\omega)^{\otimes m} \otimes \mathfrak{T}/\mathfrak{b}$ with $L(m\omega, \mathfrak{b}) = \mathscr{L}(V(\omega)^{\otimes m})$ endowed with the $\mathfrak{g}$-module structure defined by

$$(x \otimes P)(v \otimes Q) = \sum_{j=0}^{m-1} P(a_j) x^{(j)} v \otimes PQ, \qquad D(v \otimes Q) = v \otimes \frac{dQ}{dt},$$

for all $x \in \mathfrak{g}_0$, $v \in V(\omega)^{\otimes m}$, $P, Q \in \mathbb{C}[t, t^{-1}]$, where $x^{(j)}$ is defined by (1.2). Set $H(L(m\omega, \mathfrak{b})) = \bigoplus_{n \in \mathbb{Z}} L(m\omega, \mathfrak{b})_{m\omega + n\delta}$.

LEMMA. *Suppose that $a_0, \dots, a_{m-1}$ are distinct.*

1. *Assume that $v \in L(m\omega, \mathfrak{b})_{m\omega - k\alpha + n\delta}$ for some $0 \le k \le m$ and for some $n \in \mathbb{Z}$ satisfies $x_\alpha(r)v = 0$, $\forall r \in \mathbb{Z}$. Then $k = 0$. In particular every non-zero submodule of $L(m\omega, \mathfrak{b})$ meets $H(L(m\omega, \mathfrak{b}))$.*

2. *$H(L(m\omega, \mathfrak{b}))$ generates $L(m\omega, \mathfrak{b})$.*

Proof. Let $\mathscr{I}(m,k)$ be a set of all sets of k distinct integers between 0 and $m-1$.

1. We can write

$$v = \sum_{I \in \mathscr{I}(m,k)} c_I v_I \otimes t^n,$$

where

$$v_I := x_{-\alpha}^{(i_0)} \cdots x_{-\alpha}^{(i_{k-1})}\left(v_\omega^{\otimes m}\right) \tag{2.9}$$

(notice that the factor $x_{-\alpha}^{(i_j)}$ just replaces v_ω by $v_{-\omega}$ in the i_jth place). By hypothesis,

$$x_\alpha(r)v = \sum_{s=0}^{m-1} a_s^r \sum_{I \in \mathscr{I}(m,k)} c_I x_\alpha^{(s)}(v_I) \otimes t^{n+r} = 0.$$

Since the a_s are distinct, the matrix $(a_s^r)_{r,s=0}^{m-1}$ is invertible and so

$$\sum_{I \in \mathscr{I}(m,k)} c_I x_\alpha^{(s)}(v_I) = 0,$$

for all $s = 0, \dots, m-1$. Observe that $x_\alpha^{(s)} v_I = 0$ if $s \notin I$ while $x_\alpha^{(s)} v_I = v_{I \setminus \{s\}}$ if $s \in I$. We conclude that $\sum_{I \in \mathscr{I}(m,k):\, s \in I} c_I v_{I \setminus \{s\}} = 0$. The $v_{I \setminus \{s\}}$ are linearly independent; hence $c_I = 0$ if $s \in I$. Therefore, $c_I \neq 0$ implies that $s \notin I$. This forces $k = 0$.

2. Set $L = U(\mathfrak{g})H(L(m\omega, \mathfrak{b}))$ and suppose that $v_I \otimes t^n \in L$ for all $I \in \mathscr{I}(m,k)$, $n \in \mathbb{Z}$. Then for any $I \in \mathscr{I}(m,k)$ fixed

$$x_{-\alpha}(r)\left(v_I \otimes t^{n-r}\right) = \sum_{j \notin I} a_j^r v_{I \cup \{j\}} \otimes t^n, \qquad r = 0, \dots, m-k-1.$$

Since the a_j are distinct, the matrix $\{a_j^r : j \notin I,\ r = 0, \dots, m-k-1\}$ is invertible and so $v_{I'} \otimes t^n \in L$ for all $n \in \mathbb{Z}$ and all $I' \in \mathscr{I}(m, k+1)$ such that $I' = I \cup \{j\}$ for some $j \notin I$. Then $v_{I'} \otimes t^n \in L$ for all $I' \in \mathscr{I}(m, k+1)$ and $n \in \mathbb{Z}$; hence by induction on k, $L = L(m\omega, \mathfrak{b})$. ∎

3. MODULES OF TYPE $(m\ell, m)$

3.1

For all $n \in \mathbb{Z}$ we denote by $\bar{n}$ the unique representative, $0 \le \bar{n} \le m-1$, of the residue class of n modulo m.

From now on we assume that $\mathfrak{b}$ is generated by $t_i - a_i t_0$, $i = 1, \ldots, m\ell - 1$, where $a_{rm+s} = b_r \zeta^s$, $0 \le r < \ell$, $0 \le s < m$, and ζ is an mth complex primitive root of unity. We say that $\mathfrak{b}$ is *generic* if, for all $i \neq j$, b_i/b_j is not an mth root of unity and

$$p_{\pm rm}(b_0, \ldots, b_{\ell-1}) \neq 0, \qquad r = 1, \ldots, \ell.$$

These conditions ensure that $a_0, \ldots, a_{m\ell-1}$ are distinct and m is maximal. We always assume that $\mathfrak{b}$ is generic. We proceed to describe simple submodules of $L(m\ell\omega, \mathfrak{b})$ and compute their characters.

LEMMA. *A submodule M^r of $L(m\ell\omega, \mathfrak{b})$ with $\mathfrak{b}$ generic generated by $v_\omega^{\otimes m\ell} \otimes t^r$ is simple. In particular, if $m = 1$, then $L(\ell\omega, \mathfrak{b}) = M^0$ is simple.*

Proof. Observe that M^r is generated by $H(M^r)$, which is a simple $U(\mathfrak{p}^+)$-module. Let N be a non-zero $U(\mathfrak{g})$-submodule of M^r. By Lemma 2.5, $H(N) = N \cap H(M^r)$ is a non-zero $U(\mathfrak{p}^+)$-submodule of $H(M^r)$. Then $H(N) = H(M^r)$ and $M^r \subset N$. ▮

Obviously, M^r is a simple bounded module of type $(m\ell, m)$. Moreover, by Proposition 2.2 any simple bounded module $\overline{V}$ of type $(m\ell, m)$ is isomorphic to $M^r \otimes \mathbb{C}_{a\delta}$ for some $r \in \{0, \ldots, m-1\}$ and $a \in \mathbb{C}$, where $\mathbb{C}_\lambda : \lambda \in \mathfrak{h}^*$ denotes the one-dimensional $\mathfrak{g}$-module with highest weight λ. In particular, $M^r \cong M^0 \otimes \mathbb{C}_{r\delta}$; hence $\dim M^r_{\nu - r\delta} = \dim M^0_\nu$, for all $\nu \in \mathfrak{h}^*$. The module M^r is generic if the corresponding ideal is.

3.2

Evidently,

$$M^0_{m\ell\omega - k\alpha + n\delta} = \sum_{s \equiv 0 \pmod m} U(\mathfrak{g}^-)_{-k\alpha + (n-s)\delta} v_\omega^{\otimes m\ell} \otimes t^s.$$

For any $\boldsymbol{\lambda} = (\lambda_0, \ldots, \lambda_{k-1}) \in \mathbb{Z}^k$, set

$$U_{\boldsymbol{\lambda}} := x_{-\alpha}(\lambda_0) \cdots x_{-\alpha}(\lambda_{k-1}) \in U(\mathfrak{g}^-),$$

where $\lambda_i \in \mathbb{Z}$. Since $U(\mathfrak{g}^-)$ is a commutative algebra, we may assume that $\lambda_0 \ge \cdots \ge \lambda_{k-1}$. Slightly abusing standard terminology, we call $\boldsymbol{\lambda} = (\lambda_0, \ldots, \lambda_{k-1}) \in \mathbb{Z}^k$ such that $\lambda_0 \ge \cdots \ge \lambda_{k-1}$ and $|\boldsymbol{\lambda}| := \lambda_0 + \cdots + \lambda_{k-1} = n$ a *partition* of n of length $\le k$. We denote the set of partitions of n of length $\le k$ by $\mathscr{P}_k(n)$. Then, by the PBW theorem $\{U_{\boldsymbol{\lambda}} : \boldsymbol{\lambda} \in \mathscr{P}_k(n)\}$ is a basis of $U(\mathfrak{g}^-)_{-k\alpha + n\delta}$. Define

$$\mathscr{P}_k(\bar{n}, m) = \coprod_{j \in \mathbb{Z}} \mathscr{P}_k(\bar{n} + jm).$$

Then $\{U_{\boldsymbol{\lambda}} v_{\omega}^{\otimes m\ell} \otimes t^{n-|\boldsymbol{\lambda}|} : \boldsymbol{\lambda} \in \mathscr{P}_k(\bar{n}, m)\}$ span $M^0_{m\ell\omega - k\alpha + n\delta}$. For each $\boldsymbol{\lambda} \in \mathscr{P}_k(\bar{n}, m)$,

$$U_{\boldsymbol{\lambda}} v_{\omega}^{\otimes m\ell} \otimes t^{n-|\boldsymbol{\lambda}|} = \sum_{I \in \mathscr{I}(m\ell, k)} m_{\boldsymbol{\lambda}}(a_I) v_I \otimes t^n, \tag{3.1}$$

where $a_I = (a_{i_0}, \dots, a_{i_{k-1}})$, v_I is defined by (2.9) and $m_{\boldsymbol{\lambda}}$ denotes the monomial symmetric function in k variables

$$m_{\boldsymbol{\lambda}}(z_0, \dots, z_{k-1}) = \sum_{\sigma \in S_k} z_0^{\lambda_{\sigma(0)}} \cdots z_{k-1}^{\lambda_{\sigma(k-1)}}.$$

For each $(k_0, \dots, k_{\ell-1}) \in \mathbb{N}^{\ell}$ such that $0 \le k_j \le m$ and $k_0 + \cdots + k_{\ell-1} = k$ and each ℓ-tuple $(I^{(0)}, \dots, I^{(\ell-1)})$, $I^{(j)} \in \mathscr{I}(m, k_j)$, set $v_{I^{(0)}, \dots, I^{(\ell-1)}} := v_I$, where

$$I = \Big(i_0^{(0)}, \dots, i_{k_0-1}^{(0)}, i_0^{(1)} + m, \dots, i_{k_1-1}^{(1)} + m, \dots,$$
$$i_0^{(\ell-1)} + (\ell-1)m, \dots, i_{k_{\ell-1}-1}^{(\ell-1)} + (\ell-1)m\Big).$$

Then (3.1) can be written as

$$U_{\boldsymbol{\lambda}} v_{\omega}^{\otimes m\ell} \otimes t^{n-|\boldsymbol{\lambda}|} = \sum_{\substack{k_0 + \cdots + k_{\ell-1} = k \\ 0 \le k_j \le m}} \sum_{\substack{I^{(0)} \in \mathscr{I}(m, k_0) \\ \vdots \\ I^{(\ell-1)} \in \mathscr{I}(m, k_{\ell-1})}} m_{\boldsymbol{\lambda}}\Big(b_0 \zeta^{I^{(0)}}, \dots, b_{\ell-1} \zeta^{I^{(\ell-1)}}\Big) \cdot v_{I^{(0)}, \dots, I^{(\ell-1)}} \otimes t^n, \tag{3.2}$$

where $b_j \zeta^{I^{(j)}} = (b_j \zeta^{i_0^{(j)}}, \dots, b_j \zeta^{i_{k_j-1}^{(j)}})$.

3.3

Fix q, $0 < q < m$. Denote by G the group of mth complex roots of unity which is isomorphic to $\mathbb{Z}/m\mathbb{Z}$. Fix $0 < q < m$ and let $\tilde{G}^q$ be a set of all subsets of k elements of G. Write $\mathbf{g} = (g_0, \dots, g_{q-1})$. The group G acts of $\tilde{G}^q$ by $g\mathbf{g} = (gg_0, \dots, gg_{q-1})$. Observe that there is a bijection between $\tilde{G}^q$ and the set $\mathscr{I}(m, q)$ defined in 2.5; namely, $I \in \mathscr{I}(m, q)$ corresponds to $\mathbf{g} = \zeta^I$.

Let us go back to (3.2). With each $I^{(j)} = (i_0, \dots, i_{k_j-1}) \in \mathscr{I}(m, k_j)$ we associate $\mathbf{g}^{(j)} = \zeta^{I^{(j)}} \in \tilde{G}^{k_j}$. Then (3.2) becomes

$$U_{\boldsymbol{\lambda}} v_{\omega}^{\otimes m\ell} \otimes t^{n-|\boldsymbol{\lambda}|} = \sum_{\substack{k_0 + \cdots + k_{\ell-1} = k \\ 0 \le k_j \le m}} \sum_{\substack{\mathbf{g}^{(0)} \in \tilde{G}^{k_0} \\ \vdots \\ \mathbf{g}^{(\ell-1)} \in \tilde{G}^{k_{\ell-1}}}} m_{\boldsymbol{\lambda}}\Big(b_0 \mathbf{g}^{(0)}, \dots, b_{\ell-1} \mathbf{g}^{(\ell-1)}\Big) \cdot v_{(\mathbf{g}^{(0)}, \dots, \mathbf{g}^{(\ell-1)})} \otimes t^n, \tag{3.3}$$

where $b_j\mathbf{g}^{(j)} = (b_j g_0^{(j)}, \dots, b_j g_{k_j-1}^{(j)})$. Notice that the values of $m_{\boldsymbol{\lambda}}$ are well defined. Set

$$\tilde{G}^{(k_0,\dots,k_{\ell-1})} = \tilde{G}^{k_0} \times \cdots \times \tilde{G}^{k_{\ell-1}}$$

and denote by $(\mathbf{g}^{(0)}, \dots, \mathbf{g}^{(\ell-1)})$ a generic element of $\tilde{G}^{(k_0,\dots,k_{\ell-1})}$. Extend our G-action to the entire $\tilde{G}^{(k_0,\dots,k_{\ell-1})}$ and write $\tilde{G}^{(k_0,\dots,k_{\ell-1})}$ as a disjoint union of G-orbits

$$\tilde{G}^{(k_0,\dots,k_{\ell-1})} = \coprod_{(\mathbf{g}^{(0)},\dots,\mathbf{g}^{(\ell-1)}) \in \mathscr{O}^{(k_0,\dots,k_{\ell-1})}} G\big(\mathbf{g}^{(0)}, \dots, \mathbf{g}^{(\ell-1)}\big),$$

where $\mathscr{O}^{(k_0,\dots,k_{\ell-1})}$ stands for some fixed set of representatives.

Since $m_{\boldsymbol{\lambda}}$ is a homogeneous function of degree $|\boldsymbol{\lambda}| \equiv \bar{n} \pmod m$, (3.3) reduces to

$$U_{\boldsymbol{\lambda}} v_\omega^{\otimes m\ell} \otimes t^{n-|\boldsymbol{\lambda}|} = \sum_{\substack{k_0+\cdots \\ +k_{\ell-1}=k \\ 0\le k_j \le m}} \sum_{(\mathbf{g}^{(0)},\dots,\mathbf{g}^{(\ell-1)}) \in \mathscr{O}^{(k_0,\dots,k_{\ell-1})}} m_{\boldsymbol{\lambda}}\big(b_0\mathbf{g}^{(0)}, \dots, b_{\ell-1}\mathbf{g}^{(\ell-1)}\big)$$
$$\cdot\, v_{(\mathbf{g}^{(0)},\dots,\mathbf{g}^{(\ell-1)}),|\boldsymbol{\lambda}|} \otimes t^n,$$

where

$$v_{(\mathbf{g}^{(0)},\dots,\mathbf{g}^{(\ell-1)}),|\boldsymbol{\lambda}|} := \sum_{g \in G/\mathrm{Stab}_G(\mathbf{g}^{(0)},\dots,\mathbf{g}^{(\ell-1)})} g^{|\boldsymbol{\lambda}|} v_{g(\mathbf{g}^{(0)},\dots,\mathbf{g}^{(\ell-1)})}.$$

For $\bar{n}$ and $\boldsymbol{\lambda} \in \mathscr{P}_k(\bar{n}, m)$ fixed, the $\{v_{(\mathbf{g}^{(0)},\dots,\mathbf{g}^{(\ell-1)}),|\boldsymbol{\lambda}|} : (\mathbf{g}^{(0)}, \dots, \mathbf{g}^{(\ell-1)}) \in \mathscr{O}^{(k_0,\dots,k_{\ell-1})}\}$ are linearly independent. Moreover,

$$v_{(\mathbf{g}^{(0)},\dots,\mathbf{g}^{(\ell-1)}),|\boldsymbol{\lambda}|} = v_{(\mathbf{g}^{(0)},\dots,\mathbf{g}^{(\ell-1)}),|\boldsymbol{\mu}|}$$

for all $\boldsymbol{\lambda}, \boldsymbol{\mu} \in \mathscr{P}_k(\bar{n}, m)$. Therefore, $\dim M^0_{m\ell\omega - k\alpha + n\delta}$ equals the rank of the matrix

$$M(n,k) := \big\{m_{\boldsymbol{\lambda}}\big(b_0\mathbf{g}^{(0)}, \dots, b_{\ell-1}\mathbf{g}^{(\ell-1)}\big)\big\}$$

whose rows are indexed by $\boldsymbol{\lambda} \in \mathscr{P}_k(\bar{n}, m)$ and columns are indexed by $(\mathbf{g}^{(0)}, \dots, \mathbf{g}^{(\ell-1)}) \in \mathscr{O}^{(k_0,\dots,k_{\ell-1})}$ for all possible $(k_0, \dots, k_{\ell-1}) \in \mathbb{N}^\ell$ such that $k_0 + \cdots + k_{\ell-1} = k$ and $0 \le k_j \le m$. Although the matrix $M(n,k)$ depends on the choices of $\mathscr{O}^{(k_0,\dots,k_{\ell-1})}$, its rank does not. Indeed, replacing $(\mathbf{g}^{(0)}, \dots, \mathbf{g}^{(\ell-1)}) \in \mathscr{O}^{(k_0,\dots,k_{\ell-1})}$ by $g(\mathbf{g}^{(0)}, \dots, \mathbf{g}^{(\ell-1)})$ for some $g \in G$, we just multiply the corresponding column of $M(n,k)$ by $g^{\bar{n}}$, which is a non-zero complex number. Thus, we may identify $\mathscr{O}^{(k_0,\dots,k_{\ell-1})}$ with the left coset $G \backslash \tilde{G}^{(k_0,\dots,k_{\ell-1})}$.

4. THE CHARACTER FORMULAE

4.1

Fix k, $0 < k \leq m\ell$, and $(k_0, \dots, k_{\ell-1}) \in \mathbb{N}^\ell$, $k_0 + \cdots + k_{\ell-1} = k$ and $0 \leq k_j \leq m$. Denote the order of an element $g \in G$ by $\operatorname{ord} g$. Since G is abelian, $g \in \operatorname{Stab}_G(\mathbf{g}^{(0)}, \dots, \mathbf{g}^{(\ell-1)})$, $(\mathbf{g}^{(0)}, \dots, \mathbf{g}^{(\ell-1)}) \in \tilde{G}^{(k_0, \dots, k_{\ell-1})}$, implies that $g^{k_r} = 1$; hence $\operatorname{ord} g \mid k_r$, $r = 0, \dots, \ell - 1$. Observe that if $g_1, g_2 \in \operatorname{Stab}_G(\mathbf{g}^{(0)}, \dots, \mathbf{g}^{(\ell-1)})$, $\operatorname{ord} g_i = n_i$, then $\operatorname{Stab}_G(\mathbf{g}^{(0)}, \dots, \mathbf{g}^{(\ell-1)})$ also contains an element of order $\operatorname{lcm}(n_1, n_2)$. For all $g \in G$, set

$$\mathscr{F}'_{(k_0, \dots, k_{\ell-1})}(g) = \left\{ \left(\mathbf{g}^{(0)}, \dots, \mathbf{g}^{(\ell-1)}\right) \in \tilde{G}^{(k_0, \dots, k_{\ell-1})} : \right. \\ \left. g \in \operatorname{Stab}_G\left(\mathbf{g}^{(0)}, \dots, \mathbf{g}^{(\ell-1)}\right) \right\}.$$

Observe that $\mathscr{F}'_{(k_0, \dots, k_{\ell-1})}(g) = \mathscr{F}'_{(k_0, \dots, k_{\ell-1})}(h)$ if g, h are of the same order. Therefore it makes sense to define, for all $d \mid m$,

$$\mathscr{F}'_{(k_0, \dots, k_{\ell-1})}(d) := \left\{ \left(\mathbf{g}^{(0)}, \dots, \mathbf{g}^{(\ell-1)}\right) \in \tilde{G}^{(k_0, \dots, k_{\ell-1})} : \right. \\ \left. g \in \operatorname{Stab}_G\left(\mathbf{g}^{(0)}, \dots, \mathbf{g}^{(\ell-1)}\right), \operatorname{ord} g = d \right\}.$$

Observe that $\mathscr{F}'_{(k_0, \dots, k_{\ell-1})}(1) = \tilde{G}^{(k_0, \dots, k_{\ell-1})}$ and $\mathscr{F}'_{(k_0, \dots, k_{\ell-1})}(m) = \varnothing$ unless $k = m\ell$. Now define $\mathscr{F}_{(k_0, \dots, k_{\ell-1})}(d) \subset \mathscr{F}'_{(k_0, \dots, k_{\ell-1})}(d)$ as follows:

$$\mathscr{F}_{(k_0, \dots, k_{\ell-1})}(d) := \mathscr{F}'_{(k_0, \dots, k_{\ell-1})}(d) \Big\backslash \bigcup_{d' > d} \mathscr{F}'_{(k_0, \dots, k_{\ell-1})}(d').$$

It is a set of elements of $\mathscr{F}'_{(k_0, \dots, k_{\ell-1})}(d)$ which are stabilized only by $g \in G$ of order d and their powers. We say that an element $(\mathbf{g}^{(0)}, \dots, \mathbf{g}^{(\ell-1)}) \in \mathscr{O}^{(k_0, \dots, k_{\ell-1})}$ is of length r if the corresponding G-orbit in $\tilde{G}^{(k_0, \dots, k_{\ell-1})}$ contains precisely r different elements. Notice that the G-orbit of each $(\mathbf{g}^{(0)}, \dots, \mathbf{g}^{(\ell-1)}) \in \mathscr{F}_{(k_0, \dots, k_{\ell-1})}(d)$ contains exactly m/d different elements. Therefore, $\mathscr{O}^{(k_0, \dots, k_{\ell-1})}$ contains $\frac{d}{m} |\mathscr{F}_{(k_0, \dots, k_{\ell-1})}(d)|$ elements of length m/d.

From now on we use the convention that $\binom{x}{y} = 0$ if either of x, y is not an integer. Let $\mu(n)$, $n \in \mathbb{N}^+$, denote the Möbius function. By definition, $\mu(n) = 0$ if n is divisible by a square and $\mu(n) = (-1)^r$ if n is a product of r distinct primes.

LEMMA. *For any* $0 < d \leq m$,

$$\left|\mathscr{F}_{(k_0, \dots, k_{\ell-1})}(d)\right| = \sum_{dd' \mid m} \mu(d') \binom{m/dd'}{k_0/dd'} \cdots \binom{m/dd'}{k_{\ell-1}/dd'}.$$

Proof. Observe that

$$\mathscr{F}'_{(k_0,\dots,k_{\ell-1})}(d) \cap \mathscr{F}'_{(k_0,\dots,k_{\ell-1})}(d') = \mathscr{F}'_{(k_0,\dots,k_{\ell-1})}(\mathrm{lcm}(d,d')).$$

In particular, $\mathscr{F}'_{(k_0,\dots,k_{\ell-1})}(p^2d) \subset \mathscr{F}'_{(k_0,\dots,k_{\ell-1})}(pd)$ for p prime. Then, by the inclusion–exclusion principle,

$$\begin{aligned} \left|\mathscr{F}_{(k_0,\dots,k_{\ell-1})}(d)\right| &= \left|\mathscr{F}'_{(k_0,\dots,k_{\ell-1})}(d)\right| - \sum_{p \text{ prime}} \left|\mathscr{F}'_{(k_0,\dots,k_{\ell-1})}(pd)\right| \\ &\quad + \sum_{p \neq q \text{ prime}} \left|\mathscr{F}'_{(k_0,\dots,k_{\ell-1})}(pqd)\right| - \cdots \\ &= \sum_{d'd \mid m} \mu(d') \left|\mathscr{F}'_{(k_0,\dots,k_{\ell-1})}(d'd)\right|. \end{aligned}$$

Let us prove that

$$\left|\mathscr{F}'_{(k_0,\dots,k_{\ell-1})}(d)\right| = \binom{m/d}{k_0/d} \cdots \binom{m/d}{k_{\ell-1}/d}.$$

If $d \nmid k_r$, $r = 0,\dots,\ell-1$, or $d \nmid m$ then $\mathscr{F}'_{(k_0,\dots,k_{\ell-1})}(d) = \emptyset$. Suppose that $s_r = k_r/d$ are integers and take $(\mathbf{g}^{(0)},\dots,\mathbf{g}^{(\ell-1)}) \in \mathscr{F}'_{(k_0,\dots,k_{\ell-1})}(d)$. Then $g = \zeta^{m/d}$ stabilizes $(\mathbf{g}^{(0)},\dots,\mathbf{g}^{(\ell-1)})$. Evidently, $\mathbf{g}^{(r)} = \coprod_{i=0}^{s_r-1} \langle g \rangle g_i^{(r)}$ as a set, for some distinct $g_0^{(r)},\dots,g_{s_r-1}^{(r)} \in G$. Since an orbit $\langle g \rangle g_i^{(r)}$ is uniquely determined by any of its elements, we may choose $g_i^{(r)} = \zeta^{\alpha_i^{(r)}}$ where $0 \le \alpha_i^{(r)} \le \frac{m}{d} - 1$. Therefore, one can associate with $(\mathbf{g}^{(0)},\dots, \mathbf{g}^{(\ell-1)}) \in \mathscr{F}'_{(k_0,\dots,k_{\ell-1})}(d)$ a unique ℓ-tuple $I = (I^{(0)},\dots,I^{(\ell-1)}) \in \mathscr{A}(m/d, s_0) \times \cdots \times \mathscr{A}(m/d, s_{\ell-1})$, where $I^{(r)} = (\alpha_0^{(r)},\dots,\alpha_{s_r-1}^{(r)})$. Conversely, any $I = (I^{(0)},\dots,I^{(\ell-1)}) \in \mathscr{A}(m/d, s_0) \times \cdots \times \mathscr{A}(m/d, s_{\ell-1})$ yields a unique element of $\mathscr{F}'_{(k_0,\dots,k_{\ell-1})}(d)$, namely

$$\begin{aligned} &\big(\zeta^{I^{(0)}}, \zeta^{m/d}\zeta^{I^{(0)}},\dots,\zeta^{m-m/d}\zeta^{I^{(0)}},\dots, \\ &\quad \zeta^{I^{(\ell-1)}}, \zeta^{m/d}\zeta^{I^{(\ell-1)}},\dots,\zeta^{m-m/d}\zeta^{I^{(\ell-1)}}\big). \end{aligned}$$

Therefore, there is a bijection between $\mathscr{F}'_{(k_0,\dots,k_{\ell-1})}(d)$ and $\mathscr{A}(m/d, s_0) \times \cdots \times \mathscr{A}(m/d, s_{\ell-1})$. ∎

4.2

Now we can compute the number of non-zero columns in the matrix $M(n,k)$ which gives an upper bound for $\dim M^0_{m\ell\omega - k\alpha + n\delta}$.

LEMMA. *Let* $(\mathbf{g}^{(0)},\dots,\mathbf{g}^{(\ell-1)})$ *be an element of* $\mathscr{O}^{(k_0,\dots,k_{\ell-1})}$ *of length* m/d, $d\,|\,m$. *If* $d \nmid \bar{n}$ *then all homogeneous symmetric functions* f *in* k *variables,* $\deg f \equiv \bar{n} \pmod{m}$ *vanish at* $(b_0\mathbf{g}^{(0)},\dots,b_{\ell-1}\mathbf{g}^{(\ell-1)}) \in \mathbb{C}^k$. *If* $d\,|\,\bar{n}$ *then there exists a homogeneous symmetric polynomial* f, $\deg f \equiv \bar{n}$ (mod m), *such that* $f(b_0\mathbf{g}^{(0)},\dots,b_{\ell-1}\mathbf{g}^{(\ell-1)}) \neq 0$.

Proof. Assume that $(\mathbf{g}^{(0)},\dots,\mathbf{g}^{(\ell-1)}) \in \mathscr{F}_{(k_0,\dots,k_{\ell-1})}(d)$. Then $g\mathbf{g}^{(r)} = \mathbf{g}^{(r)}$, $r = 0,\dots,\ell-1$, for any $g \in G$ of order d; hence, for any homogeneous symmetric function f in k variables of degree n,

$$f\big(b_0\mathbf{g}^{(0)},\dots,b_{\ell-1}\mathbf{g}^{(\ell-1)}\big) = f\big(b_0 g\mathbf{g}^{(0)},\dots,b_{\ell-1} g\mathbf{g}^{(\ell-1)}\big)$$
$$= g^{\bar{n}} f\big(b_0\mathbf{g}^{(0)},\dots,b_{\ell-1}\mathbf{g}^{(\ell-1)}\big).$$

Therefore, if $d \nmid \bar{n}$, then $f(b_0\mathbf{g}^{(0)},\dots,b_{\ell-1}\mathbf{g}^{(\ell-1)}) = 0$.

Define a map $\tilde{G}^{(k_0,\dots,k_{\ell-1})} \to \mathbb{C}[X]$ by

$$\big(\mathbf{g}^{(0)},\dots,\mathbf{g}^{(\ell-1)}\big) \mapsto P_{(\mathbf{g}^{(0)},\dots,\mathbf{g}^{(\ell-1)})}(X) := \prod_{r=0}^{\ell-1}\prod_{i=0}^{k_r-1}\big(X - b_r g_i^{(r)}\big).$$

Since for all $i \neq j$, b_i/b_j is not an mth root of unity, $P_{g(\mathbf{g}^{(0)},\dots,\mathbf{g}^{(\ell-1)})}(X) = P_{(\mathbf{g}^{(0)},\dots,\mathbf{g}^{(\ell-1)})}(X)$ if and only if $g \in \mathrm{Stab}_G(\mathbf{g}^{(0)},\dots,\mathbf{g}^{(\ell-1)})$. On the other hand, that equality holds if and only if

$$e_r\big(b_0 g\mathbf{g}^{(0)},\dots,b_{\ell-1} g\mathbf{g}^{(\ell-1)}\big) = g^r e_r\big(b_0\mathbf{g}^{(0)},\dots,b_{\ell-1} g^{(\ell-1)}\big)$$
$$= e_r\big(b_0\mathbf{g}^{(0)},\dots,b_{\ell-1}\mathbf{g}^{(\ell-1)}\big),$$

for all r, $1 \le r \le k$. Thus, $g \in G$ stabilizes $(\mathbf{g}^{(0)},\dots,\mathbf{g}^{(\ell-1)})$ if and only if

$$e_r\big(b_0\mathbf{g}^{(0)},\dots,b_{\ell-1}\mathbf{g}^{(\ell-1)}\big) = 0, \qquad 1 \le r \le k, \operatorname{ord} g \nmid r.$$

Suppose that $d\,|\,n$ and $(\mathbf{g}^{(0)},\dots,\mathbf{g}^{(\ell-1)}) \in \mathscr{F}_{(k_0,\dots,k_{\ell-1})}(d)$. Then there exists $u \le k/d$, such that $\gcd(u,m) = 1$ and $e_{ud}(b_0\mathbf{g}^{(0)},\dots,b_{\ell-1}\mathbf{g}^{(\ell-1)}) \neq 0$. Indeed, since $(\mathbf{g}^{(0)},\dots,\mathbf{g}^{(\ell-1)})$ is stabilized by $g \in G$ of order d,

$$e_r\big(b_0\mathbf{g}^{(0)},\dots,b_{\ell-1}\mathbf{g}^{(\ell-1)}\big) = 0, \qquad \text{unless } d\,|\,r.$$

If, however $e_{rd}(b_0\mathbf{g}^{(0)},\dots,b_{\ell-1}\mathbf{g}^{(\ell-1)}) = 0$ unless $d'\,|\,r$ for some $d'\,|\,m$, then $g \in \mathrm{Stab}_G(\mathbf{g}^{(0)},\dots,\mathbf{g}^{(\ell-1)})$ for all g of order $d'd$, which is a contradiction. Yet u is a multiplicative unit of $\mathbb{Z}/m\mathbb{Z}$; hence there exists $0 < v \le m - 1$ such that $uv \equiv 1 \pmod{m}$. Then $f(x_1,\dots,x_k) := e_{ud}(x_1,\dots,x_k)^{\bar{n}v/d}$ is the desired homogeneous symmetric polynomial. ∎

COROLLARY. *The number $N_k(n)$ of non-zero columns in $M(n,k)$ equals*

$$N_k(n) = \frac{1}{m}\sum_{d\mid m}\phi_n(d)\binom{m\ell/d}{k/d}, \tag{4.1}$$

where

$$\phi_n(d) = \mu(d/\gcd(d,n))\frac{\phi(d)}{\phi(d/\gcd(d,n))}, \tag{4.2}$$

and $\phi(d)$ is the Euler function. In particular, $\dim M^0_{m\ell\omega-k\alpha+n\delta} \le N_k(\bar{n})$.

Proof. By Lemma 4.2, if $(\mathbf{g}^{(0)},\dots,\mathbf{g}^{(\ell-1)}) \in \mathscr{O}^{(k_0,\dots,k_{\ell-1})}$ is of length m/d, then a homogeneous symmetric polynomial F of degree $\bar{n} \pmod m$ not vanishing at $(b_0\mathbf{g}^{(0)},\dots,b_{\ell-1}\mathbf{g}^{(\ell-1)})$ exists if and only if $d\mid n$. Since F can be written as a linear combination of $m_{\boldsymbol{\lambda}} : \boldsymbol{\lambda} \in \mathscr{P}_k(\bar{n},m)$ we conclude that a column of $M(n,k)$ corresponding to $(\mathbf{g}^{(0)},\dots,\mathbf{g}^{(\ell-1)})$ contains a non-zero entry if and only if $d\mid n$. By Lemma 4.1, the number $N_k(n)$ of non-zero columns in $M(n,k)$ equals

$$\begin{aligned}
N_k(n) &= \sum_{\substack{k_0+\cdots+k_{\ell-1}=k\\ 0\le k_r\le m}} \sum_{d\mid m,n} \frac{d}{m}\left|\mathscr{F}_{(k_0,\dots,k_{\ell-1})}(d)\right| \\
&= \sum_{\substack{k_0+\cdots+k_{\ell-1}=k\\ 0\le k_r\le m}} \frac{1}{m}\sum_{d\mid n,\, dd'\mid m} d\mu(d')\prod_{r=0}^{l-1}\binom{m/dd'}{k_r/dd'} \\
&= \frac{1}{m}\sum_{\substack{k_0+\cdots+k_{\ell-1}=k\\ 0\le k_r\le m}} \sum_{d\mid m}\prod_{r=0}^{l-1}\binom{m/d}{k_r/d} d\sum_{d'\mid d,\, d\mid d'n}\frac{\mu(d')}{d'}.
\end{aligned} \tag{4.3}$$

Recall the identity

$$\phi(d) = d\sum_{d'\mid d}\frac{\mu(d')}{d'}. \tag{4.4}$$

Motivated by (4.4), we define

$$\phi_n(d) := d\sum_{d'\mid d,\, d\mid d'n}\frac{\mu(d')}{d'}.$$

Then (4.3) yields

$$N_k(n) = \frac{1}{m}\sum_{d\,|\,m}\phi_n(d)\sum_{\substack{k_0+\cdots+k_{\ell-1}=k\\ 0\le k_r\le m}}\prod_{r=0}^{l-1}\binom{m/d}{k_r/d}$$
$$= \frac{1}{m}\sum_{d\,|\,m}\phi_n(d)\binom{m\ell/d}{k/d}.$$

Denote the right-hand side of (4.2) by $f(n,d)$. Observe that $d'\,|\,d$, $d\,|\,d'n$ implies that $d/\gcd(d,n)\,|\,d'$. If $d/\gcd(d,n)$ is divisible by a square, then so is d'; hence $\mu(d')=0$. Thus, in that case $\phi_n(d)=0=f(n,d)$. Furthermore, suppose that $d = p_1\cdots p_r\gcd(d,n) = p_1^{\alpha_1}\cdots p_r^{\alpha_r}d_0$, where $\gcd(d_0, p_1,\dots,p_r)=1$. Then $d'\,|\,d$, $d\,|\,d'n$ if and only if $d' = p_1\cdots p_r d_0'$, $d_0'\,|\,\gcd(d,n)$ and we may assume that $d_0'\,|\,d_0$ for otherwise $\mu(d')=0$. Recall that μ is a multiplicative function and $\phi(p^\alpha)=p^{\alpha-1}(p-1)$ for any prime p. Then

$$\phi_n(d) = d\sum_{d_0'\,|\,d_0}\frac{\mu(p_1\cdots p_r d_0')}{p_1\cdots p_r d_0'}$$
$$= p_1^{\alpha_1-1}\cdots p_r^{\alpha_r-1}d_0\sum_{d'\,|\,d_0}\frac{\mu(p_1\cdots p_r)\mu(d')}{d'}$$
$$= (-1)^r p_1^{\alpha_1-1}\cdots p_r^{\alpha_r-1}\phi(d_0) = (-1)^r\frac{\phi(p_1^{\alpha_1})\cdots\phi(p_r^{\alpha_r})\phi(d_0)}{(p_1-1)\cdots(p_r-1)}$$
$$= (-1)^r\frac{\phi(d)}{(p_1-1)\cdots(p_r-1)} = f(n,d),$$

where we used (4.4) and the multiplicative property of the Euler function. Finally, if $d\,|\,n$ then $\phi_n(d)=\phi(d)=f(n,d)$ by (4.4). ∎

Suppose that $\ell=1$. Then the number $N_k(n)$ coincides, if at least one of m and k is odd, with the number $(n)_k$ of solutions (cf. [2, p. 87]) of

$$n\equiv x_1+\cdots+x_k \qquad (\mathrm{mod}\, m),\ 0\le x_1<\cdots<x_k<m.$$

4.3

Observe that $\phi_n(1)=1$ for all n; hence $\sum_{n=0}^{m-1}\phi_n(1)=m$.

LEMMA. *Let $d>1$ be a divisor of m. Then*

$$\sum_{n=0}^{m-1}\phi_n(d) = 0. \tag{4.5}$$

Proof. By definition (4.2) of $\phi_n(d)$,

$$\sum_{n=0}^{m-1} \phi_n(d) = \sum_{\gcd(n,d)=d} \phi(d) - \sum_{\substack{\gcd(n,d)=d/p \\ p \text{ prime}}} \frac{\phi(d)}{p-1} + \sum_{\substack{\gcd(n,d)=d/pq \\ p\neq q \text{ primes}}} \frac{\phi(d)}{(p-1)(q-1)} - \cdots. \tag{4.6}$$

In all the sums in the right-hand side n runs through the set $\{0,\dots,m-1\}$. Let $q_1,\dots,q_s$ be distinct prime divisors of d. Evidently, $(-1)^s\phi(d)/(q_1-1)\cdots(q_s-1)$ appears in (4.6) with the coefficient that equals the cardinality of a set

$$\{0 \le n \le m-1 : \gcd(n,d) = d/(q_1 \cdots q_s)\};$$

that is, by the inclusion–exclusion principle,

$$\frac{m}{d/q_1\cdots q_s} - \sum_{1\le i\le s} \frac{m}{d/q_1\cdots \hat{q}_i \cdots q_s} + \sum_{1\le i<j\le s} \frac{m}{d/q_1 \cdots \hat{q}_i \cdots \hat{q}_j \cdots q_s} - \cdots = \frac{m}{d}(q_1-1)\cdots(q_s-1).$$

Therefore, the right-hand side of (4.6) boils down to

$$\frac{m\phi(d)}{d}\left(1 - \binom{r}{1} + \binom{r}{2} - \cdots\right) = 0,$$

where r is the number of distinct prime factors of d. ∎

4.4

Now we are able to complete the proof of Proposition 1.3.

Proof of Proposition 1.3. Retain the notation of 2.1–2.3 and 3.1. There exist $m, \ell \in \mathbb{N}^+$ such that $M \cong \overline{V}$, where $\overline{V}$ is a generic simple bounded module of type $(m\ell, m)$. Since $\overline{V} \cong M^0 \otimes \mathbb{C}_{a\delta}$ for some $a \in \mathbb{C}$, $\operatorname{ch} \overline{V} = e^{a\delta} \operatorname{ch} M^0$. By Lemma 3.1, the modules M^r, $r = 0,\dots,m-1$, are simple and $M^r \cap M^s = 0$, $0 \le r \ne s \le m-1$. Set $L = \bigoplus_{r=0}^{m-1} M^r \subset L(m\ell\omega, \mathfrak{b})$. Evidently, $H(L(m\ell\omega,\mathfrak{b})) \subset L$; hence, by Lemma 2.5, $L(m\ell\omega,\mathfrak{b}) \subset L$. Therefore, $L(m\ell\omega,\mathfrak{b}) = L$ and

$$\sum_{r=0}^{m-1} \dim M^r_{m\ell\omega - k\alpha + n\delta} = \dim L(m\ell\omega, \mathfrak{b})_{m\ell\omega-k\alpha+n\delta} = \binom{m\ell}{k}. \tag{4.7}$$

On the other hand,

$$\sum_{r=0}^{m-1} \dim M^r_{m\ell\omega - k\alpha + n\delta} = \sum_{r=0}^{m-1} \dim M^0_{m\ell\omega - k\alpha + (n-r)\delta} \leq \sum_{r=0}^{m-1} N_k(n-r)$$

$$= \sum_{r=0}^{m-1} N_k(r) = \frac{1}{m} \sum_{d\,|\,m} \binom{m\ell/d}{k/d} \sum_{r=0}^{m-1} \phi_r(d).$$

By Lemma 4.3, the inner sum equals zero if $d > 1$ and m if $d = 1$. Thus,

$$\sum_{r=0}^{m-1} \dim M^r_{m\ell\omega - k\alpha + n\delta} \leq \binom{m\ell}{k},$$

which, together with (4.7) and Lemma 4.2, implies (1.3). ∎

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